B2.2 Commutative Algebra

Sheet 1 — HT25

Sections 1-5

Section A

1. Let R be a ring. Show that the Jacobson radical of R coincides with the set

$$\{x \in R \mid 1 - xy \text{ is a unit for all } y \in R\}.$$

Solution: Suppose that x lies in the Jacobson radical of R. Suppose for contradiction that $1 - xy$ is not a unit for some $y \in R$. Let $\mathfrak{m}$ be a maximal ideal containing $1 - xy$. We know that $xy \in \mathfrak{m}$ since $x \in \mathfrak{m}$ and thus we conclude that $1 \in \mathfrak{m}$, a contradiction.

Suppose now that $x \in R$ and that there is a maximal ideal $\mathfrak{m}$ not containing x. Then $x + \mathfrak{m}$ is non-trivial in the field $R/\mathfrak{m}$ and hence it is a unit; thus, there is a $y \in R$ such that $xy + \mathfrak{m} = 1 + \mathfrak{m}$. In other words, $1 - xy \in \mathfrak{m}$ and so $1 - xy$ is not a unit.

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Section B

2. Let R be a ring.

   (a) Show that if $P(x) = a_0 + a_1 x + \cdots + a_k x^k \in R[x]$ is a unit of $R[x]$ then a_0 is a unit of R and a_i is nilpotent for all $i \geqslant 1$.

   (b) Show that the Jacobson radical and the nilradical of $R[x]$ coincide.

3. Let R be a ring and let $N \subseteq R$ be its nilradical. Show that the following are equivalent:

   (a) R has exactly one prime ideal.

   (b) Every element of R is either a unit or is nilpotent.

   (c) R/N is a field.

4. Let R be a ring and let $I \subseteq R$ be an ideal. Let $S = \{1 + r \mid r \in I\}$.

   (a) Show that S is a multiplicative set.

   (b) Show that the ideal generated by the image of I in R_S is contained in the Jacobson radical of R_S.

   (c) Prove the following generalisation of Nakayama's lemma:

   Lemma. *Let M be a finitely generated R-module and suppose that $IM = M$. Then there exists $r \in R$, such that $r - 1 \in I$ and $rM = 0$.*

5. Let R be a ring and let M be a finitely generated R-module. Let $\phi \colon M \to M$ be a surjective homomorphism of R-modules. Prove that ϕ is injective, and is thus an automorphism. [Hint: use ϕ to construct a structure of $R[x]$-module on M and use the previous question.]

6. Let R be a ring. Let $\mathcal{S}$ be the subset of the set of ideals of R defined as follows: an ideal I is in $\mathcal{S}$ if and only if all the elements of I are zero-divisors. Show that $\mathcal{S}$ has maximal elements (for the relation of inclusion) and that every maximal element is a prime ideal. Show that the set of zero-divisors of R is a union of prime ideals.

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Section C

7. Let R be a ring. Consider the inclusion relation on the set $\operatorname{Spec}(R)$. Show that there are minimal elements in $\operatorname{Spec}(R)$.

   Solution: Let $\mathcal{T}$ be a totally ordered subset of $\operatorname{Spec}(R)$ for the relation $\supseteq$. Note that the maximal elements for the relation $\supseteq$ are the minimal elements for the inclusion relation (which is $\subseteq$). Let $I = \bigcap_{\mathfrak{p}\in\mathcal{T}}\mathfrak{p}$. Then I is an ideal. We claim that I is prime.

   To see this, let $x, y \in R$ and suppose for contradiction that $x, y \in R \smallsetminus I$ and that $xy \in I$. By assumption there are prime ideals $\mathfrak{p}_x, \mathfrak{p}_y \in \mathcal{T}$ such that $x \notin \mathfrak{p}_x$ and $y \notin \mathfrak{p}_y$. Suppose without restriction of generality that $\mathfrak{p}_x \supseteq \mathfrak{p}_y$ (recall that $\mathcal{T}$ is totally ordered). We have $xy \in \mathfrak{p}_y$ and thus either x or y lies in $\mathfrak{p}_y$. This contradicts the fact that $x, y \notin \mathfrak{p}_y$. The ideal I thus lies in $\operatorname{Spec}(R)$ and it is a lower bound for $\mathcal{T}$. We may thus apply Zorn's lemma to conclude that there are minimal elements in $\operatorname{Spec}(R)$.

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